A posteriori discontinuous Galerkin error estimator for linear elasticity: Supplementary material

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This document contains supplementary material for the journal article "A posteriori discontinuous Galerkin error estimator for linear elasticity" [2] by Robert E. Bird, William M. Coombs and Stefano Giani. This document is not a substitution to the full journal article, for this reason, we don't re-introduce the notations and the definitions already presented in the article and we invite the reader to refer to it. The rest of the document reports the proof of reliability, Theorem 6.1 in [2], for the error estimator in more detail in order to facilitate the comprehension of the argument. In the rest of the document we refer to equations in [2] in the usual way, e.g. (1) and (2), and we refer to equations presented in this document appending an S, e.g. (S.1) and (S.2)

The error estimator in Theorem 6.1 in [2] is defined as

$$\eta_{\mathrm{err}} = \sqrt{\sum_{K\in\mathcal{T}} \left(\eta_{R,K}^2 + \eta_{J,K}^2 + \eta_{F,K}^2\right)}, \tag{S.1}$$

where the three terms under the sum are defined as

$$\begin{aligned}
\eta_{R,K}^2 &:= \frac{h_K^2}{p_K^2}\left\|\mathbf{f}_h + \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}_h)\right\|_{0,K}^2,\\
\eta_{J,K}^2 &:= \frac{1}{2}\sum_{E\in\mathcal{E}^{\mathrm{int}}(K)} \frac{\gamma^2 p_E^3}{h_E}\left\|[\![\mathbf{u}_h]\!]\right\|_{0,E}^2 + \sum_{E\in\mathcal{E}^{\mathrm{D}}(K)} \frac{\gamma^2 p_E^3}{h_E}\left\|\mathbf{u}_h - \mathbf{g}_{D,h}\right\|_{0,E}^2\\
&\quad + \sum_{E\in\mathcal{E}^{\mathrm{T}}(K)} \frac{\gamma^2 p_E^3}{h_E}\left\|\mathbf{u}_h\cdot\mathbf{n} - \mathbf{g}_{T,h}\cdot\mathbf{n}\right\|_{0,E}^2, \qquad\qquad (\mathrm{S}.2)\\
\eta_{F,K}^2 &:= \frac{1}{2}\sum_{E\in\mathcal{E}^{\mathrm{int}}(K)} \frac{h_E}{p_E}\left\|[\![\boldsymbol{\sigma}(\mathbf{u}_h)]\!]\right\|_{0,E}^2 + \sum_{E\in\mathcal{E}^{\mathrm{N}}(K)} \frac{h_E}{p_E}\left\|\boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K - \mathbf{g}_{N,h}\right\|_{0,E}^2\\
&\quad + \sum_{E\in\mathcal{E}^{\mathrm{T}}(K)} \frac{h_E}{p_E}\left\|\mathbf{t}(\mathbf{u}_h)\cdot\mathbf{n}_{\|}\right\|_{0,E}^2\ ,
\end{aligned}$$

we invite the reader to refer to Section 4 in [2] for the definitions of the quantity appearing in (S.2). Moreover the oscillation term is also the sum of different terms:

$$\mathrm{osc} := \sqrt{\mathrm{osc}_{\mathrm{glo}}^2 + \sum_{K\in\mathcal{T}} \left(\mathrm{osc}_{R,K}^2 + \mathrm{osc}_{J,K}^2 + \mathrm{osc}_{F,K}^2\right)}, \tag{S.3}$$

where

$$\begin{aligned}
\mathrm{osc}_{\mathrm{glo}}^2 &:= \|\mathbf{g}_D - \mathbf{g}_{D,h}\|_{1/2,\Gamma_D} + \|\mathbf{g}_N - \mathbf{g}_{N,h}\|_{0,\Gamma_N} + \|\mathbf{g}_T - \mathbf{g}_{T,h}\|_{1/2,\Gamma_T},\\
\mathrm{osc}_{R,K}^2 &:= \frac{h_K^2}{p_K^2}\left\|\mathbf{f} - \mathbf{f}_h\right\|_{0,K}^2,\\
\mathrm{osc}_{J,K}^2 &:= \sum_{E\in\mathcal{E}^{\mathrm{D}}(K)} \frac{\gamma^2 p_E^3}{h_E}\left\|\mathbf{g}_D - \mathbf{g}_{D,h}\right\|_{0,E}^2 + \sum_{E\in\mathcal{E}^{\mathrm{T}}(K)} \frac{\gamma^2 p_E^3}{h_E}\left\|\mathbf{g}_T\cdot\mathbf{n} - \mathbf{g}_{T,h}\cdot\mathbf{n}\right\|_{0,E}^2,\\
\mathrm{osc}_{F,K}^2 &:= \sum_{E\in\mathcal{E}^{\mathrm{N}}(K)} \frac{h_E}{p_E}\left\|\mathbf{g}_N - \mathbf{g}_{N,h}\right\|_{0,E}^2,
\end{aligned}$$

where the quantities $\mathbf{f}_D$, $\mathbf{g}_D$, $\mathbf{g}_N$ and $\mathbf{g}_T$ are the data of the model problem

$$\begin{array}{rcll}
-\nabla\cdot\boldsymbol{\sigma}(\mathbf{u}) &=& \mathbf{f} & \text{in} \quad \Omega\ ,\\
\mathbf{u} &=& \mathbf{g}_D & \text{on} \quad \Gamma_D\ ,\\
\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n} &=& \mathbf{g}_N & \text{on} \quad \Gamma_N\ ,\\
\mathbf{u}\cdot\mathbf{n} &=& \mathbf{g}_T\cdot\mathbf{n} & \text{on} \quad \Gamma_T\ ,\\
\mathbf{t}(\mathbf{u})\cdot\mathbf{n}_{\parallel} &=& 0 & \text{on} \quad \Gamma_T\ ,
\end{array} \tag{S.4}$$

in Section 2 in [2].

Theorem 1 (Reliability)**.** *Let* $\mathbf{u}$ *the exact solution and* $\mathbf{u}_h$ *the computed solution, we have that*

$$|||\mathbf{u} - \mathbf{u}_h|||_{\mathcal{T}} \le C(\eta_{\mathrm{err}} + \mathrm{osc}),$$

where C *is a positive constant independent of the mesh nor the order of the elements used.*

As explained in [2], we need to introduce an auxiliary continuous problem similar to (S.4) in order to carry out the analysis:

$$\begin{array}{rcll}
-\nabla\cdot\boldsymbol{\sigma}(\tilde{\mathbf{u}}) &=& \mathbf{f} & \text{in} \quad \Omega\\
\tilde{\mathbf{u}} &=& \mathbf{g}_{D,h} & \text{on} \quad \Gamma_D\\
\boldsymbol{\sigma}(\tilde{\mathbf{u}})\cdot\mathbf{n} &=& \mathbf{g}_{N,h} & \text{on} \quad \Gamma_N\\
\tilde{\mathbf{u}}\cdot\mathbf{n} &=& \mathbf{g}_{T,h}\cdot\mathbf{n} & \text{on} \quad \Gamma_T\\
\mathbf{t}(\tilde{\mathbf{u}})\cdot\mathbf{n}_{\parallel} &=& \mathbf{0} & \text{on} \quad \Gamma_T\ .
\end{array} \tag{S.5}$$

Since linear elasticity is a linear problem and solutions depended continuously on the data, we have

$$\|\boldsymbol{\nabla}(\mathbf{u} - \tilde{\mathbf{u}})\|_{0,\Omega} \lesssim \|\mathbf{g}_D - \mathbf{g}_{D,h}\|_{1/2,\Gamma_D} + \|\mathbf{g}_N - \mathbf{g}_{N,h}\|_{0,\Gamma_N} + \|\mathbf{g}_T - \mathbf{g}_{T,h}\|_{1/2,\Gamma_T}\ . \tag{S.6}$$

For DG methods, the construction of the upper bound for the DG norm using the error is done in two steps bounding separately the conforming and the non conforming part of the error. To this end, we define the space $V^c_{\mathbf{p}}(\mathcal{T}) \equiv V_{\mathbf{p}}(\mathcal{T}) \cap [H^1(\Omega)]^2$ which is a conforming version of the DG space. Then, we decompose the discontinuous Galerkin finite element space $V_{\mathbf{p}}(\mathcal{T}) = V^c_{\mathbf{p}}(\mathcal{T}) \bigoplus V^{\perp}_{\mathbf{p}}(\mathcal{T})$, where $V^{\perp}_{\mathbf{p}}(\mathcal{T})$ is the orthogonal complement of $V^c_{\mathbf{p}}(\mathcal{T})$ with respect to the DG norm (13) in [2]. We also define $V^c_{\mathbf{p},0}(\mathcal{T})$ which is the subspace of $V^c_{\mathbf{p}}(\mathcal{T})$ containing functions satisfying the following boundary conditions imposed strongly: $\mathbf{u} = \mathbf{0}$ on Γ_D, $\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{0}$ on Γ_N, $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_T and $\mathbf{t}(\mathbf{u}) \cdot \mathbf{n}_{\parallel} = \mathbf{0}$ on Γ_T. We also assume that there is an interpolation operator $I_{hp} : V_{\mathbf{p}}(\mathcal{T}) \to V^c_{\mathbf{p}}(\mathcal{T})$ that satisfy the following inequalities:

$$p_K^2 h_K^{-2} \|\mathbf{v} - I_{hp}\mathbf{v}\|^2_{0,K} \lesssim \|\boldsymbol{\nabla}\mathbf{v}\|^2_{0,K} \ , \tag{S.7}$$

$$\|\boldsymbol{\nabla}(\mathbf{v} - I_{hp}\mathbf{v})\|^2_{0,K} \lesssim \|\boldsymbol{\nabla}\mathbf{v}\|^2_{0,K} \ , \tag{S.8}$$

$$p_E h_E^{-1} \|\mathbf{v} - I_{hp}\mathbf{v}\|^2_{0,E} \lesssim \|\boldsymbol{\nabla}\mathbf{v}\|^2_{0,K} \ , \tag{S.9}$$

with $E \subset \partial K$.

We can then split the solution as:

$$\mathbf{u}_h - I_{hp}\tilde{\mathbf{u}} = \mathbf{u}^c_h + \mathbf{u}^r_h, \tag{S.10}$$

with $\mathbf{u}^c_h \in V^c_{\mathbf{p},0}(\mathcal{T})$ and $\mathbf{u}^r_h \in V^{\perp}_{\mathbf{p}}(\mathcal{T})$, then using the triangle inequality and (S.10), we obtain

$$\begin{aligned} |||\mathbf{u} - \mathbf{u}_h|||_{\mathcal{T}} &\leq |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{T}} + |||\tilde{\mathbf{u}} - \mathbf{u}_h|||_{\mathcal{T}} \\ &\leq |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{T}} + |||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}^c_h - \mathbf{u}^r_h|||_{\mathcal{T}} \\ &\leq |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{T}} + |||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}^c_h|||_{\mathcal{T}} + |||\mathbf{u}^r_h|||_{\mathcal{T}}. \end{aligned} \tag{S.11}$$

The first term on the rhs of (S.11) can be bounded using (S.6) and (13) noticing that $\mathbf{u} - \tilde{\mathbf{u}}$ is zero on the internal edges because they are both functions in $[H^1(\Omega)]^2$:

$$\begin{aligned} |||\mathbf{u} - \tilde{\mathbf{u}}|||^2_{\mathcal{T}} = \|\boldsymbol{\nabla}(\mathbf{u} - \tilde{\mathbf{u}})\|^2_{0,\Omega} &+ \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \left\| \mathbf{g}_D - \mathbf{g}_{D,h} \right\|^2_{0,E} \\ &+ \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \left\| (\mathbf{g}_T - \mathbf{g}_{T,h}) \cdot \mathbf{n} \right\|^2_{0,E} \lesssim \mathrm{osc}^2_{\mathrm{glo}} + \mathrm{osc}^2_{J,K} \lesssim \mathrm{osc}^2 \ . \end{aligned} \tag{S.12}$$

Then, in order to obtain the upper bound for the conforming part of the error $|||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}^c_h|||_{\mathcal{T}}$, we recognize that $[\![\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}^c_h]\!] = 0$ in the interior of the mesh because $\tilde{\mathbf{u}} \in [H^1(\Omega)]^2$ and both $I_{hp}\tilde{\mathbf{u}}$ and $\mathbf{u}^c_h$ are by construction in $V^c_{\mathbf{p}}(\mathcal{T})$. So denoting

$$\begin{aligned} D(\mathbf{u}, \mathbf{v}) := \sum_{K \in \mathcal{T}} \int_K \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) \, d\mathbf{x} &+ \sum_{E \in \mathcal{E}^{\mathrm{int}}(\mathcal{T}) \cup \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E [\![\mathbf{u}]\!] : [\![\mathbf{v}]\!] \, ds \\ &+ \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{u} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}) \, ds \ , \end{aligned} \tag{S.13}$$

we obtain from the coercivity result for the continuous case, Theorem 2.2 in [2]:

$$|||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c|||_{\mathcal{T}}^2 \lesssim D(\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c, \tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c) \ . \tag{S.14}$$

Then setting

$$\mathbf{v} = \frac{\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c}{|||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c|||_{\mathcal{T}}} \in V_{\mathbf{p},0}^c(\mathcal{T}) \ , \tag{S.15}$$

we have using (S.10)

$$\begin{aligned} |||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c|||_{\mathcal{T}} &\lesssim D(\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c, \mathbf{v}) = D(\tilde{\mathbf{u}} - \mathbf{u}_h, \mathbf{v}) + D(\mathbf{u}_h^r, \mathbf{v}) \\ &= D(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + D(\tilde{\mathbf{u}} - \mathbf{u}, \mathbf{v}) + D(\mathbf{u}_h^r, \mathbf{v}) \ . \end{aligned} \tag{S.16}$$

The term $D(\tilde{\mathbf{u}} - \mathbf{u}, \mathbf{v})$ can be bounded using the Cauchy-Schwarz inequality and (S.12) by

$$D(\tilde{\mathbf{u}} - \mathbf{u}, \mathbf{v}) \lesssim |||\mathbf{u} - \tilde{\mathbf{u}}|||_{\mathcal{T}} |||\mathbf{v}|||_{\mathcal{T}} \lesssim \text{osc } |||\mathbf{v}|||_{\mathcal{T}} \ . \tag{S.17}$$

To bound the term $D(\mathbf{u}_h^r, \mathbf{v})$ we use Cauchy-Schwarz inequality and Lemma 6.2 in [2].

$$D(\mathbf{u}_h^r, \mathbf{v}) \lesssim |||\mathbf{u}_h^r|||_{\mathcal{T}} |||\mathbf{v}|||_{\mathcal{T}} \lesssim \eta_{\text{err}} |||\mathbf{v}|||_{\mathcal{T}} \ . \tag{S.18}$$

Therefore, the remaining quantity to bound is $D(\mathbf{u} - \mathbf{u}_h, \mathbf{v})$. Firstly, we rewrite (12) as

$$D(\mathbf{u}_h, \mathbf{v}_h) + K(\mathbf{u}_h, \mathbf{v}_h) = l(\mathbf{v}_h) = l^c(\mathbf{v}_h) + l^r(\mathbf{v}_h) \ , \tag{S.19}$$

where

$$\begin{aligned} K(\mathbf{u}, \mathbf{v}) :=& - \sum_{E \in \mathcal{E}^{\text{int}}(\mathcal{T}) \cup \mathcal{E}^{\text{D}}(\mathcal{T})} \int_E \{\boldsymbol{\sigma}(\mathbf{u})\} : [\![\mathbf{v}]\!] + \{\boldsymbol{\sigma}(\mathbf{v})\} : [\![\mathbf{u}]\!] \, ds \\ & - \sum_{E \in \mathcal{E}^{\text{T}}(\mathcal{T})} \int_E (\mathbf{t}(\mathbf{u}) \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}) + (\mathbf{t}(\mathbf{v}) \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{n}) \, ds \\ l^c(\mathbf{v}) :=& \sum_{K \in \mathcal{T}} \int_K \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} \\ & + \sum_{E \in \mathcal{E}^{\text{N}}(\mathcal{T})} \int_E \mathbf{g}_N \cdot \mathbf{v} \, ds \ , \\ l^r(\mathbf{v}) :=& - \sum_{E \in \mathcal{E}^{\text{D}}(\mathcal{T})} \int_E \mathbf{g}_D \cdot \boldsymbol{\sigma}(\mathbf{v}) \cdot \mathbf{n} \, ds + \sum_{E \in \mathcal{E}^{\text{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E \mathbf{g}_D \cdot \mathbf{v} \, ds \\ & - \sum_{E \in \mathcal{E}^{\text{T}}(\mathcal{T})} \int_E (\mathbf{g}_T \cdot \mathbf{n})(\mathbf{t}(\mathbf{v}) \cdot \mathbf{n}) \, ds + \sum_{E \in \mathcal{E}^{\text{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{g}_T \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n}) \, ds \ . \end{aligned}$$

Then, since $V_{\mathbf{p},0}^c(\mathcal{T}) \in \mathcal{S}_2^{\text{CG}}$, we have for v satisfying (S.15) and using (6)

$$D(\mathbf{u}, \mathbf{v}) = a^{\text{CG}}(\mathbf{u}, \mathbf{v}) = l^{\text{CG}}(\mathbf{v}) = l^c(\mathbf{v}) \ ,$$

which leads to

$$D(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) = D(\mathbf{u},\mathbf{v}) - D(\mathbf{u}_h,\mathbf{v}) = l^c(\mathbf{v}) - D(\mathbf{u}_h,\mathbf{v}) \ .$$

Further, using (S.19) we have:

$$\begin{aligned} D(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) &= l^c(\mathbf{v}) - D(\mathbf{u}_h,\mathbf{v}) \\ &= l^c(\mathbf{v}-\mathbf{v}_h) - l^r(\mathbf{v}_h) - D(\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h) + K(\mathbf{u}_h,\mathbf{v}_h) \ . \end{aligned} \tag{S.20}$$

The final step to bound $D(\mathbf{u}-\mathbf{u}_h,\mathbf{v})$ is Lemma 6.3 in [2], which is also reported here with a more detailed proof.

Lemma 2. *Considering* $\mathbf{u}_h$ *the DG solution of problem (12) and for any continuous function* $\mathbf{v}$ *with* $\mathbf{v}_h := I_{hp}\mathbf{v}$*, we have:*

$$l^c(\mathbf{v}-\mathbf{v}_h) - l^r(\mathbf{v}_h) - D(\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h) + K(\mathbf{u}_h,\mathbf{v}_h) \lesssim (\eta_{\text{err}} + \text{osc})|||\mathbf{v}|||_{\mathcal{T}} \ .$$

Proof. Applying integration by parts to the first term in $D(\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h)$:

$$\begin{aligned} & l^c(\mathbf{v}-\mathbf{v}_h) - l^r(\mathbf{v}_h) - D(\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h) + K(\mathbf{u}_h,\mathbf{v}_h) \\ &= l^c(\mathbf{v}-\mathbf{v}_h) - l^r(\mathbf{v}_h) - \sum_{K\in\mathcal{T}} \int_K \boldsymbol{\sigma}(\mathbf{u}_h) : \boldsymbol{\epsilon}(\mathbf{v}-\mathbf{v}_h)\, d\mathbf{x} \\ &- \sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})\cup\mathcal{E}^{\text{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E [\![\mathbf{u}_h]\!] : [\![\mathbf{v}-\mathbf{v}_h]\!]\, ds \\ &- \sum_{E\in\mathcal{E}^{\text{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{u}_h\cdot\mathbf{n})(\mathbf{v}-\mathbf{v}_h)\cdot\mathbf{n}\, ds + K(\mathbf{u}_h,\mathbf{v}_h) \\ &= l^c(\mathbf{v}-\mathbf{v}_h) - l^r(\mathbf{v}_h) + \sum_{K\in\mathcal{T}} \int_K \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}_h)\cdot(\mathbf{v}-\mathbf{v}_h)\, d\mathbf{x} \\ &- \sum_{K\in\mathcal{T}} \int_{\partial K} \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_k\cdot(\mathbf{v}-\mathbf{v}_h)\, ds \\ &- \sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})\cup\mathcal{E}^{\text{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E [\![\mathbf{u}_h]\!] : [\![\mathbf{v}-\mathbf{v}_h]\!]\, ds \\ &- \sum_{E\in\mathcal{E}^{\text{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{u}_h\cdot\mathbf{n})(\mathbf{v}-\mathbf{v}_h)\cdot\mathbf{n}\, ds + K(\mathbf{u}_h,\mathbf{v}_h) \ . \end{aligned} \tag{S.21}$$

The term $\sum_{K\in\mathcal{T}} \int_{\partial K} \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot(\mathbf{v}-\mathbf{v}_h)\, ds$ can be further treated recalling the definitions of the jump $[\![\cdot]\!]$ operator and the average $\{\cdot\}$ operator, see (11) in [2]:

$$
\begin{aligned}
\sum_{K\in\mathcal{T}} \int_{\partial K} \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot(\mathbf{v}-\mathbf{v}_h)\,ds &= \sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})} \int_E \{\boldsymbol{\sigma}(\mathbf{u}_h)\} : [\![\mathbf{v}-\mathbf{v}_h]\!]\,ds \\
&+ \sum_{E\in\mathcal{E}^{\text{BC}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot(\mathbf{v}-\mathbf{v}_h)\,ds + \sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})} \int_E [\![\boldsymbol{\sigma}(\mathbf{u}_h)]\!]\cdot\{\mathbf{v}-\mathbf{v}_h\}\,ds \\
&= -\sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})} \int_E \{\boldsymbol{\sigma}(\mathbf{u}_h)\} : [\![\mathbf{v}_h]\!]\,ds + \sum_{E\in\mathcal{E}^{\text{BC}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot(\mathbf{v}-\mathbf{v}_h)\,ds \\
&+ \sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})} \int_E [\![\boldsymbol{\sigma}(\mathbf{u}_h)]\!] : \{\mathbf{v}-\mathbf{v}_h\}\,ds \ ,
\end{aligned}
$$

where in the last step we used the fact that from (S.15) $[\![\mathbf{v}]\!] = \mathbf{0}$ in the interior of the mesh. Considering also the term $K(\mathbf{u}_h, \mathbf{v}_h)$ we have:

$$
\begin{aligned}
K(\mathbf{u}_h,\mathbf{v}_h) - \sum_{K\in\mathcal{T}} \int_{\partial K} \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot(\mathbf{v}-\mathbf{v}_h)\,ds &= -\sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})} \int_E [\![\mathbf{u}_h]\!] : \{\boldsymbol{\sigma}(\mathbf{v}_h)\}\,ds \\
&- \sum_{E\in\mathcal{E}^{\text{D}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot\mathbf{v}_h + \mathbf{u}_h\cdot\boldsymbol{\sigma}(\mathbf{v}_h)\cdot\mathbf{n}_K\,ds \\
&- \sum_{E\in\mathcal{E}^{\text{T}}(\mathcal{T})} \int_E (\mathbf{t}(\mathbf{u}_h)\cdot\mathbf{n})(\mathbf{v}_h\cdot\mathbf{n}) + (\mathbf{u}_h\cdot\mathbf{n})(\mathbf{t}(\mathbf{v}_h)\cdot\mathbf{n})\,ds \\
&- \sum_{E\in\mathcal{E}^{\text{BC}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot(\mathbf{v}-\mathbf{v}_h)\,ds - \sum_{E\in\mathcal{E}^{\text{int}}(\mathcal{T})} \int_E [\![\boldsymbol{\sigma}(\mathbf{u}_h)]\!] : \{\mathbf{v}-\mathbf{v}_h\}\,ds \ .
\end{aligned}
$$

In view of this, (S.21) becomes

$$
\begin{aligned}
& l^c(\mathbf{v}-\mathbf{v}_h) - l^r(\mathbf{v}_h) - D(\mathbf{u}_h, \mathbf{v}-\mathbf{v}_h) + K(\mathbf{u}_h, \mathbf{v}_h) \\
&= \sum_{K\in\mathcal{T}} \int_K \mathbf{f}\cdot(\mathbf{v}-\mathbf{v}_h)\, d\mathbf{x} + \sum_{E\in\mathcal{E}^{\mathrm{N}}(\mathcal{T})} \int_E \mathbf{g}_N\cdot(\mathbf{v}-\mathbf{v}_h)\, ds \\
&\quad + \sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \mathbf{g}_D\cdot\boldsymbol{\sigma}(\mathbf{v}_h)\cdot\mathbf{n}\, ds - \sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E \mathbf{g}_D\cdot\mathbf{v}_h\, ds \\
&\quad + \sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E (\mathbf{g}_T\cdot\mathbf{n})(\mathbf{t}(\mathbf{v}_h)\cdot\mathbf{n})\, ds - \sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{g}_T\cdot\mathbf{n})(\mathbf{v}_h\cdot\mathbf{n})\, ds \\
&\quad + \sum_{K\in\mathcal{T}} \int_K \nabla\cdot\boldsymbol{\sigma}(\mathbf{u}_h)\cdot(\mathbf{v}-\mathbf{v}_h)\, d\mathbf{x} - \sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})\cup\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E [\![\mathbf{u}_h]\!] : [\![\mathbf{v}-\mathbf{v}_h]\!]\, ds \\
&\quad - \sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{u}_h\cdot\mathbf{n})(\mathbf{v}-\mathbf{v}_h)\cdot\mathbf{n}\, ds - \sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \int_E [\![\mathbf{u}_h]\!] : \{\boldsymbol{\sigma}(\mathbf{v}_h)\}\, ds \\
&\quad - \sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot\mathbf{v}_h + \mathbf{u}_h\cdot\boldsymbol{\sigma}(\mathbf{v}_h)\cdot\mathbf{n}_K\, ds \\
&\quad - \sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E (\mathbf{t}(\mathbf{u}_h)\cdot\mathbf{n})(\mathbf{v}_h\cdot\mathbf{n}) + (\mathbf{u}_h\cdot\mathbf{n})(\mathbf{t}(\mathbf{v}_h)\cdot\mathbf{n})\, ds \\
&\quad - \sum_{E\in\mathcal{E}^{\mathrm{BC}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K\cdot(\mathbf{v}-\mathbf{v}_h)\, ds - \sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \int_E [\![\boldsymbol{\sigma}(\mathbf{u}_h)]\!] : \{\mathbf{v}-\mathbf{v}_h\}\, ds\ ,
\end{aligned}
$$

which can be split in four terms defined as:

$$
\begin{aligned}
T_1 &:= \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{v}_h) + \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_h) \cdot (\mathbf{v} - \mathbf{v}_h) \, d\mathbf{x} \ , \\
T_2 &:= - \sum_{E \in \mathcal{E}^{\mathrm{int}}(\mathcal{T}) \cup \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E [\![\mathbf{u}_h]\!] : [\![\mathbf{v} - \mathbf{v}_h]\!] \, ds - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{u}_h \cdot \mathbf{n})(\mathbf{v} - \mathbf{v}_h) \cdot \mathbf{n} \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E \mathbf{g}_D \cdot \mathbf{v}_h \, ds - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma p_E^2}{h_E} \int_E (\mathbf{g}_T \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n}) \, ds \ , \\
T_3 &:= - \sum_{E \in \mathcal{E}^{\mathrm{int}}(\mathcal{T})} \int_E [\![\boldsymbol{\sigma}(\mathbf{u}_h)]\!] : \{\mathbf{v} - \mathbf{v}_h\} \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot (\mathbf{v} - \mathbf{v}_h) \, ds - \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot \mathbf{v}_h \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{N}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot (\mathbf{v} - \mathbf{v}_h) \, ds + \sum_{E \in \mathcal{E}^{\mathrm{N}}(\mathcal{T})} \int_E \mathbf{g}_N \cdot (\mathbf{v} - \mathbf{v}_h) \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot (\mathbf{v} - \mathbf{v}_h) \, ds - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E (\mathbf{t}(\mathbf{u}_h) \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n}) \, ds \\
T_4 &:= - \sum_{E \in \mathcal{E}^{\mathrm{int}}(\mathcal{T})} \int_E [\![\mathbf{u}_h]\!] \cdot \{\boldsymbol{\sigma}(\mathbf{v}_h)\} \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \mathbf{u}_h \cdot \boldsymbol{\sigma}(\mathbf{v}_h) \cdot \mathbf{n}_K \, ds + \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \mathbf{g}_D \cdot \boldsymbol{\sigma}(\mathbf{v}_h) \cdot \mathbf{n}_K \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E (\mathbf{u}_h \cdot \mathbf{n})(\mathbf{t}(\mathbf{v}_h) \cdot \mathbf{n}) \, ds + \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E (\mathbf{g}_T \cdot \mathbf{n})(\mathbf{t}(\mathbf{v}_h) \cdot \mathbf{n}) \, ds \ ,
\end{aligned}
$$

The rest of the proof consists in bounding each term separately. To bound T_1, we use the Cauchy-Schwarz inequality, (S.7) with $\mathbf{v}_h := I_{hp}\mathbf{v}$:

$$
\begin{aligned}
T_1 &\leq \left[\left(\sum_{K \in \mathcal{T}} \frac{h_K^2}{p_K^2} \|\mathbf{f}_h + \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_h)\|_{0,K}^2 \right)^{1/2} + \left(\sum_{K \in \mathcal{T}} \frac{h_K^2}{p_K^2} \|\mathbf{f} - \mathbf{f}_h\|_{0,K}^2 \right)^{1/2} \right] \\
&\quad \times \left(\sum_{K \in \mathcal{T}} \frac{p_K^2}{h_K^2} \|\mathbf{v} - \mathbf{v}_h\|_{0,K}^2 \right)^{1/2} \\
&\lesssim \left[\left(\sum_{K \in \mathcal{T}} \eta_{R,K}^2 \right)^{1/2} + \left(\sum_{K \in \mathcal{T}} \mathrm{osc}_{R,K}^2 \right)^{1/2} \right] |||\mathbf{v}|||_{\mathcal{T}} \ .
\end{aligned}
$$

To bound T_2 we notice that from (S.15) $\mathbf{v} \in V^c_{\mathbf{p},0}(\mathcal{T})$, therefore $\mathbf{v} = \mathbf{0}$ on Γ_D and $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$

on Γ_T. This together with Cauchy-Schwarz inequality and (S.9) with $\mathbf{v}_h := I_{hp}\mathbf{v}$ lead to

$$
\begin{aligned}
T_2 \leq & \Bigg[\Bigg(\sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \frac{\gamma^2 p_E^3}{h_E}\|[\![\mathbf{u}_h]\!]\|_{0,E}^2 + \sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma^2 p_E^3}{h_E}\|\mathbf{u}_h - \mathbf{g}_{D,h}\|_{0,E}^2 \\
& + \sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma^2 p_E^3}{h_E}\|\mathbf{u}_h\cdot\mathbf{n} - \mathbf{g}_{T,h}\cdot\mathbf{n}\|_{0,E}^2\Bigg)^{1/2} \\
& + \Bigg(\sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma^2 p_E^3}{h_E}\|\mathbf{g}_D - \mathbf{g}_{D,h}\|_{0,E}^2 + \sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma^2 p_E^3}{h_E}\|\mathbf{g}_T\cdot\mathbf{n} - \mathbf{g}_{T,h}\cdot\mathbf{n}\|_{0,E}^2\Bigg)^{1/2}\Bigg] \\
& \times \Bigg(\sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \frac{p_E}{h_E}\|[\![\mathbf{v}-\mathbf{v}_h]\!]\|_{0,E}^2 + \sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{p_E}{h_E}\|\mathbf{v}-\mathbf{v}_h\|_{0,E}^2 \\
& + \sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{p_E}{h_E}\|(\mathbf{v}-\mathbf{v}_h)\cdot\mathbf{n}\|_{0,E}^2\Bigg)^{1/2} \\
\lesssim & \Bigg[\Bigg(\sum_{K\in\mathcal{T}}\eta_{J,K}^2\Bigg)^{1/2} + \Bigg(\sum_{K\in\mathcal{T}}\mathrm{osc}_{J,K}^2\Bigg)^{1/2}\Bigg] |||\mathbf{v}|||_{\mathcal{T}} \ .
\end{aligned}
$$

To bound T_3 in the interior of the mesh, we use again the Cauchy-Schwarz inequality and (S.9) with $\mathbf{v}_h := I_{hp}\mathbf{v}$:

$$
\begin{aligned}
T_3|_{\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \leq & \Bigg(\sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \frac{h_E}{p_E}\|[\![\boldsymbol{\sigma}(\mathbf{u}_h)]\!]\|_{0,E}^2\Bigg)^{1/2} \times \Bigg(\sum_{E\in\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \frac{p_E}{h_E}\|\{\mathbf{v}-\mathbf{v}_h\}\|_{0,E}^2\Bigg)^{1/2} \\
\lesssim & \Bigg(\sum_{K\in\mathcal{T}}\eta_{F,K}^2\Bigg)^{1/2} |||\mathbf{v}|||_{\mathcal{T}} \ .
\end{aligned}
$$

In a similar way T_3 is bounded on the Neumann portion of boundary:

$$
\begin{aligned}
T_3|_{\mathcal{E}^{\mathrm{N}}(\mathcal{T})} \leq & \Bigg[\Bigg(\sum_{E\in\mathcal{E}^{\mathrm{N}}(\mathcal{T})} \frac{h_E}{p_E}\|\boldsymbol{\sigma}(\mathbf{u}_h)\cdot\mathbf{n}_K - \mathbf{g}_{N,h}\|_{0,E}^2\Bigg)^{1/2} \\
& + \Bigg(\sum_{E\in\mathcal{E}^{\mathrm{N}}(\mathcal{T})} \frac{h_E}{p_E}\|\mathbf{g}_N - \mathbf{g}_{N,h}\|_{0,E}^2\Bigg)^{1/2}\Bigg] \\
& \times \Bigg(\sum_{E\in\mathcal{E}^{\mathrm{N}}(\mathcal{T})} \frac{p_E}{h_E}\|\mathbf{v}-\mathbf{v}_h\|_{0,E}^2\Bigg)^{1/2} \\
\lesssim & \Bigg[\Bigg(\sum_{K\in\mathcal{T}}\eta_{F,K}^2\Bigg)^{1/2} + \Bigg(\sum_{K\in\mathcal{T}}\mathrm{osc}_{F,K}^2\Bigg)^{1/2}\Bigg] |||\mathbf{v}|||_{\mathcal{T}} \ .
\end{aligned}
$$

On the Dirichlet portion of boundary, the term T_3 is null since from (S.15) $\mathbf{v} \in V^c_{\mathbf{p},0}(\mathcal{T})$ and so $\mathbf{v} = \mathbf{0}$ on Γ_D:

$$
\begin{aligned}
T_3|_{\mathcal{E}^{\mathrm{D}}(\mathcal{T})} &= - \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot (\mathbf{v} - \mathbf{v}_h)\, ds - \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot \mathbf{v}_h \, ds \\
&= - \sum_{E \in \mathcal{E}^{\mathrm{D}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot \mathbf{v} \, ds = 0 \; .
\end{aligned}
$$

On the traction portion of boundary for T_3 we use again $\mathbf{v} \in V^c_{\mathbf{p},0}(\mathcal{T})$ from (S.15) that implies $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$:

$$
\begin{aligned}
T_3|_{\mathcal{E}^{\mathrm{T}}(\mathcal{T})} &= - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot (\mathbf{v} - \mathbf{v}_h)\, ds - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E (\mathbf{t}(\mathbf{u}_h) \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n}) \, ds \\
&= - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot \mathbf{n}_{\|} (\mathbf{v} - \mathbf{v}_h) \cdot \mathbf{n}_{\|} \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot \mathbf{n} (\mathbf{v} - \mathbf{v}_h) \cdot \mathbf{n} \, ds \\
&\quad - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E (\mathbf{t}(\mathbf{u}_h) \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n}) \, ds \\
&= - \sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \int_E \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot \mathbf{n}_{\|} (\mathbf{v} - \mathbf{v}_h) \cdot \mathbf{n}_{\|} \, ds \\
&\leq \left(\sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{h_E}{p_E} \| \boldsymbol{\sigma}(\mathbf{u}_h) \cdot \mathbf{n}_K \cdot \mathbf{n}_{\|} \|^2_{0,E} \right)^{1/2} \times \left(\sum_{E \in \mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{p_E}{h_E} \| \mathbf{v} - \mathbf{v}_h \|^2_{0,E} \right)^{1/2} \\
&\lesssim \left(\sum_{K \in \mathcal{T}} \eta^2_{F,K} \right)^{1/2} |||\mathbf{v}|||_{\mathcal{T}} \; .
\end{aligned}
$$

To bound T_4 in the interior of the mesh, we use (4), the Cauchy-Schwarz inequality and the standard hp-version of the trace inequality [7]:

$$
\begin{aligned}
T_4|_{\mathcal{E}^{\mathrm{int}}(\mathcal{T})} &\lesssim \left(\sum_{E \in \mathcal{E}^{\mathrm{int}}(\mathcal{T})} \frac{\gamma^2 p_E^2}{h_E} \| [\![\mathbf{u}_h]\!] \|^2_{0,E} \right)^{1/2} \times \left(\sum_{E \in \mathcal{E}^{\mathrm{int}}(\mathcal{T})} \frac{h_E}{p_E^2} \| \{ \boldsymbol{\nabla} \mathbf{v}_h \} \|^2_{0,E} \right)^{1/2} \\
&\lesssim \left(\sum_{K \in \mathcal{T}} \eta^2_{J,K} \right)^{1/2} \left(\sum_{K \in \mathcal{T}} \| \boldsymbol{\nabla} \mathbf{v}_h \|^2_{0,K} \right)^{1/2} \; .
\end{aligned}
\tag{S.22}
$$

Then, using (S.8) we have

$$
\| \boldsymbol{\nabla} \mathbf{v}_h \|^2_{0,K} \lesssim \| \boldsymbol{\nabla} (\mathbf{v} - \mathbf{v}_h) \|^2_{0,K} + \| \boldsymbol{\nabla} \mathbf{v} \|^2_{0,K} \lesssim \| \boldsymbol{\nabla} \mathbf{v} \|^2_{0,K} \; ,
$$

which can be used in (S.22) to obtain

$$T_4|_{\mathcal{E}^{\mathrm{int}}(\mathcal{T})} \lesssim \left(\sum_{K\in\mathcal{T}} \eta_{J,K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}} \|\boldsymbol{\nabla}\mathbf{v}\|_{0,K}^2\right)^{1/2} \lesssim \left(\sum_{K\in\mathcal{T}} \eta_{J,K}^2\right)^{1/2} |||\mathbf{v}|||_{\mathcal{T}} \ .$$

In a similar way T_4 is bounded on the Dirichlet portion of boundary:

$$\begin{aligned} T_4|_{\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \lesssim & \left[\left(\sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma^2 p_E^2}{h_E}\|\mathbf{u}_h - \mathbf{g}_{D,h}\|_{0,E}^2\right)^{1/2}\right. \\ & \left. + \sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{\gamma^2 p_E^2}{h_E}\|\mathbf{g}_D - \mathbf{g}_{D,h}\|_{0,E}^2\right)^{1/2}\right] \\ & \times \left(\sum_{E\in\mathcal{E}^{\mathrm{D}}(\mathcal{T})} \frac{h_E}{p_E^2}\|\boldsymbol{\nabla}\mathbf{v}_h\|_{0,E}^2\right)^{1/2} \\ \lesssim & \left[\left(\sum_{K\in\mathcal{T}} \eta_{J,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}} \mathrm{osc}_{J,K}^2\right)^{1/2}\right] |||\mathbf{v}|||_{\mathcal{T}} \ , \end{aligned}$$

and also the traction portion of boundary for T_4:

$$\begin{aligned} T_4|_{\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \lesssim & \left[\left(\sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma^2 p_E^2}{h_E}\|\mathbf{u}_h\cdot\mathbf{n} - \mathbf{g}_{T,h}\cdot\mathbf{n}\|_{0,E}^2\right)^{1/2}\right. \\ & \left. + \left(\sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{\gamma^2 p_E^2}{h_E}\|\mathbf{g}_T\cdot\mathbf{n} - \mathbf{g}_{T,h}\cdot\mathbf{n}\|_{0,E}^2\right)^{1/2}\right] \\ & \times \left(\sum_{E\in\mathcal{E}^{\mathrm{T}}(\mathcal{T})} \frac{h_E}{p_E^2}\|\boldsymbol{\nabla}\mathbf{v}_h\cdot\mathbf{n}\|_{0,E}^2\right)^{1/2} \\ \lesssim & \left[\left(\sum_{K\in\mathcal{T}} \eta_{J,K}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}} \mathrm{osc}_{J,K}^2\right)^{1/2}\right] |||\mathbf{v}|||_{\mathcal{T}} \ . \end{aligned}$$

The statement of the lemma is a consequence of all the above bounds.

□

The upper bound for (S.16) comes from the application of Lemma 2, (S.17) and (S.18)

$$|||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c|||_{\mathcal{T}} \lesssim (\eta_{\mathrm{err}} + \mathrm{osc})|||\mathbf{v}|||_{\mathcal{T}} \ .$$

From (S.15) we have that $|||\mathbf{v}|||_{\mathcal{T}} = 1$, therefore

$$|||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c|||_{\mathcal{T}} \lesssim \eta_{\mathrm{err}} + \mathrm{osc} \ . \tag{S.23}$$

Now the proof of Theorem 1 can be achieved from (S.11)

$$|||\mathbf{u}-\mathbf{u}_h|||_{\mathcal{T}} \lesssim |||\mathbf{u}-\tilde{\mathbf{u}}|||_{\mathcal{T}} + |||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c|||_{\mathcal{T}} + |||\mathbf{u}_h^r|||_{\mathcal{T}} \ ,$$

using (S.12) and Lemma 6.2 in [2] to obtain

$$|||\mathbf{u}-\mathbf{u}_h|||_{\mathcal{T}} \lesssim \mathrm{osc} + |||\tilde{\mathbf{u}} - I_{hp}\tilde{\mathbf{u}} - \mathbf{u}_h^c|||_{\mathcal{T}} + \eta_{\mathrm{err}} \ .$$

Finally, using (S.23) we obtain

$$|||\mathbf{u}-\mathbf{u}_h|||_{\mathcal{T}} \lesssim \mathrm{osc} + \eta_{\mathrm{err}} \ .$$

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